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Closed-Loop System Identification by Residual Whitening

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The accuracy of the system model during the identification process for dynamic systems under closed-loop operation is studied. The simulation results show that most of the model error comes from the approximated system description by using a finite difference model called the autogressive with exogeneous input model. Then the residual whitening method is proposed to improve the model accuracy. An iterative procedure for minimizing and whitening the residual of the autogressive moving average with exogeneous input (ARMAX) model is provided. Numerical simulations of a highly unstable, large-gap magnetic suspension system are presented to validate the proposed residual whitening method.

Introduction

SYSTEM identification is a process of determining the model structure and parameters of a system. Through system identification, one can derive a mathematical model for the system and design a controller to reach the desired performance. Most existing system identification methods[1–4] apply for stable systems without requiring feedback control for identification purpose. For identifying marginally stable or unstable systems, feedback control is required to ensure overall system stability. Recently, Huang et al.[5] proposed an indirect method to identify unstable systems with a known feedback controller under closed-loop operation. It includes four steps in the identification process: 1) Estimate the closed-loop autogressive with exogeneous input (ARX) model coefficient matrices. 2) Compute the closed-loop system and Kalman filter Markov parameters from the estimated coefficient matrices. 3) Compute the open-loop system and Kalman filter Markov parameters from the closed-loop system and Kalman filter Markov parameters and controller Markov parameters calculated from the known controller dynamics. 4) Realize the open-loop system matrices and Kalman filter gain from the open-loop system and Kalman filter Markov parameters by using the eigensystem realization algorithm (ERA).[2] The ARX model is one of the simplest input–output descriptions and is basically a linear finite difference equation. In some literature it is also called an error equation model. The first objective of this paper is to find out where the model error is induced in the aforementioned identification processes, so that the model accuracy can be improved. It turns out the truncation of the approximated ARX coefficient matrices generates the major error.

Recently, Phan et al.[6,7] and Juang and Phan[8] introduced a method to improve observer/Kalman filter identification (OKID) by residual whitening for an open-loop system, and the resultant residual is minimized, uncorrelated with the input and output data, and also white. This method uses a direct system identification approach and usually applies for open-loop systems. The second objective of this paper is to extend this work to indirectly identify systems under closed-loop operations by whitening the residual during the estimation of the closed-loop ARX model coefficient matrices. In this approach, however, one uses a closed-loop autogressive moving average with exogeneous input (ARMAX) model that contains the noise dynamics by the moving averaging terms instead of using the ARX model. Finally, an example of identifying an unstable, large-gap magnetic suspension system is provided with numerical simulations to illustrate the improvement of the proposed indirect identification method.

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Algorithm for Closed-Loop System Identification

A finite dimensional, linear, discrete-time, time-invariant stochastic system can be expressed as

$$\boldsymbol{x}_{k+1} = A\boldsymbol{x}_k + B\boldsymbol{u}_k + w_k \tag{1}$$

$$\boldsymbol{y}_k = C\boldsymbol{x}_k + v_k \tag{2}$$

where $\boldsymbol{x}_k \in R^{n\times 1}$ is the state vector, $\boldsymbol{u} \in R^{ni\times 1}$ is the signal input vector, and $\boldsymbol{y} \in R^{no\times 1}$ is the output vector; $[A,\ B,\ C]$ are the system matrices. The sequences of process noise $w \in R^{n\times 1}$ and measurement noise $v \in R^{no\times 1}$ are assumed white, Gaussian, and zero mean. The noises w_k and v_k are also assume uncorrelated with covariance Q and R, respectively.

Equations (1) and (2) can also be expressed as[2]

$$\hat{\boldsymbol{x}}_{k+1} = A\hat{\boldsymbol{x}}_k + B\boldsymbol{u}_k + AK\varepsilon_k \tag{3}$$

$$\boldsymbol{y}_k = C\hat{\boldsymbol{x}}_k + \varepsilon_k \tag{4}$$

where $\varepsilon_k \in R^{no\times 1}$ is the output residual with $\varepsilon_k = \boldsymbol{y}_k - \hat{\boldsymbol{y}}_k$. It is zero mean, white, and Gaussian; $\hat{\boldsymbol{x}}_k$ and $\hat{\boldsymbol{y}}_k$ are the estimated state and output, respectively. $K \in R^{n\times no}$ is the steady-state Kalman filter gain. The existence of K is guaranteed if the system is detectable and $(A,\ Q^{1/2})$ is stabilizable. The Kalman filter in this case is equivalent to the Wiener filter.

For unstable systems, a dynamic feedback controller is added to ensure the system's stability. A dynamic feedback controller can be expressed as[5]

$$\boldsymbol{p}_{k+1} = A_d\boldsymbol{p}_k + B_d\boldsymbol{y}_k \tag{5}$$

$$\boldsymbol{u}_k = C_d\boldsymbol{p}_k + D_d\boldsymbol{y}_k + \boldsymbol{r}_k \tag{6}$$

where $\boldsymbol{p}_k$, $\boldsymbol{r}_k$, $[A_d\ \ B_d\ \ C_d\ \ D_d]$ are the controller state, reference input to the closed-loop system, and state-space matrices of the controller, respectively. Taking Eqs. (3–6) together, the augmented closed-loop system dynamics becomes

$$\eta_{k+1} = A_c\eta_k + B_c r_k + A_c K_c \varepsilon_k \tag{7}$$

$$\boldsymbol{y}_k = C_c\eta_k + \varepsilon_k \tag{8}$$

where

$$\eta_k = \begin{bmatrix} \hat{x}_k \\ p_k \end{bmatrix}, \qquad A_c = \begin{bmatrix} A + BD_dC & BC_d \\ B_dC & A_d \end{bmatrix}, \qquad B_c = \begin{bmatrix} B \\ 0 \end{bmatrix}$$

$$A_cK_c = \begin{bmatrix} AK + BD_d \\ B_d \end{bmatrix}, \qquad C_c = [C \quad 0]$$

It can be shown that[5]

$$y_k = \sum_{i=1}^{q} c_i y_{k-i} + \sum_{i=1}^{q} d_i r_{k-i} + \varepsilon_k \tag{9}$$

where

$$c_i = C_c\bar{A}_c^{i-1}A_cK_c, \qquad d_i = C_c\bar{A}_c^{i-1}B_c, \qquad i = 1, 2, \ldots, q$$

$$\bar{A}_c = A_c - A_cK_cC_c \tag{10}$$

Equation (9) is the closed-loop ARX model that directly represents the relationship between the reference input and output of the closed-loop system. Here, q is the closed-loop system ARX model order, and c_i and d_i are the closed-loop ARX model parameters used to find the closed-loop system and Kalman filter Markov parameters. Suppose that there are N given data points of y_k and r_k, $k = 0, 1, \ldots, N-1$, the indirect closed-loop system identification includes the following four steps.[5]

1) Estimate the coefficient matrices of the ARX model from the closed-loop input/output data. The batch least-squares solution for estimating the parameters c_i and d_i is

$$\theta_{\text{clid}} = YV^T(VV^T)^{-1} \tag{11}$$

where

$$Y = [y_0 \quad y_1 \quad \ldots \quad y_q \quad \ldots \quad y_{N-1}]$$

$$\theta_{\text{clid}} = \begin{bmatrix} d_1 & c_1 & \cdots & d_q & c_q \end{bmatrix}$$

$$V = \begin{bmatrix} 0 & r_0 & \ldots & r_{q-1} & \ldots & r_{N-2} \\ 0 & y_0 & \ldots & y_{q-1} & \ldots & y_{N-2} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & r_0 & \ldots & r_{N-q-1} \\ 0 & 0 & 0 & y_0 & \ldots & y_{N-q-1} \end{bmatrix}$$

2) Compute the closed-loop system and Kalman filter Markov parameters from the estimated coefficient matrices c_i and d_i of the closed-loop ARX model:

$$Y_c(k) = d_k + \sum_{i=1}^{k} c_i Y_c(k-i) \tag{12}$$

$$N_c(k) = \sum_{i=1}^{k} c_i N_c(k-i) \tag{13}$$

Also note that $Y_c(0) = 0$, and $N_c(0) = I$, and $c_i = d_i = 0$ when $i > q$.

3) By using the closed-loop system Markov parameters $Y_c(k)$ and the Kalman filter Markov parameters $N_c(k)$, as well as the known controller Markov parameters $Y_d(k)$, one can derive the open-loop system Markov parameters $Y(k) = CA^{k-1}B$ and Kalman filter Markov parameters $N(k) = CA^{k-1}AK$:

$$Y(j) = Y_c(j) - \sum_{k=1}^{j}\sum_{i=1}^{k} Y(i)Y_d(k-i)Y_c(j-k) \tag{14}$$

$$N(j) = N_c(j) - \sum_{k=1}^{j}\sum_{i=1}^{k} Y(i)Y_d(k-i)N_c(j-k) \tag{15}$$

Note that $Y_c(0) = 0$, and $N_c(0) = I$.

4) Use the ERA method to realize the open-loop system matrices and recover the open-loop Kalman filter gain.

Error Development During the Identification Process

The identification process and its error detection for the Markov parameters are described in Fig. 1. Given the analytical system, $[A \quad B \quad C]$, one can produce y and r, from which the ARX model coefficients, closed-loop, and then open-loop Markov parameters and the identified system matrices $[A_i \quad B_i \quad C_i]$ can be computed by using the four steps described in the last section. From the analytical model and the identified model one can calculate directly the open-loop Markov parameters, so that one gets three sets of open-loop Markov parameters and three sets of closed-loop Markov parameters. Comparing the Markov parameters obtained from the analytical model with the identified Markov parameters, one can quantify the total error of the system identification process. Comparing the Markov parameters calculated from the ARX model, with the Markov parameters from the identified system, one can get the error composed by the noise influence and the system identification steps following the computation of the ARX model. Several comparisons between identified and calculated Markov parameters are presented. The system used for the simulations is a large-gap magnetic suspension system for validating the indirect identification methods.[5]

Three sets of closed-loop Markov parameters can be computed using the diagram shown in Fig. 2. To ensure that an efficient identification is utilized the ARX model order and the data length have to be optimized first. The optimal model order is determined by using a variable ARX model order and noise variance. The error deviation of the first 30 Markov parameters is computed for every data pair. Therefore, the error between the first 30 true and reconstructed open-loop system Markov parameters is defined as

$$\sum_{i=1}^{30} \frac{\|\hat{C}\hat{A}^{i-1}\hat{B} - CA^{i-1}B\|_F}{\|CA^{i-1}B\|_F} \tag{16}$$

where the caret denotes the reconstructed ones, the F norm is defined as $\|X\|_F = \sqrt{[\text{tr}(X^T \times X)]}$, and X is a matrix.

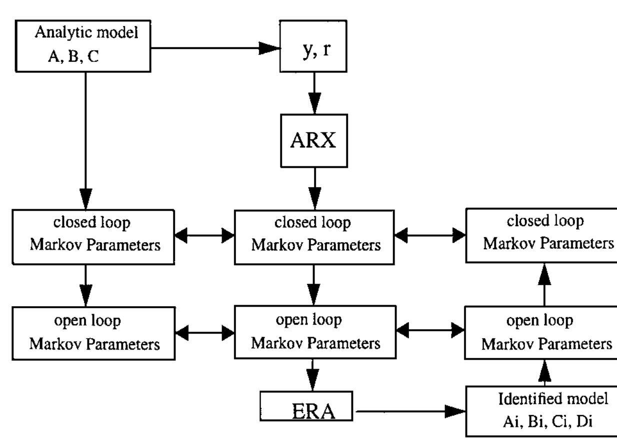

Fig. 1 Block diagram for computing model errors during the identification process.

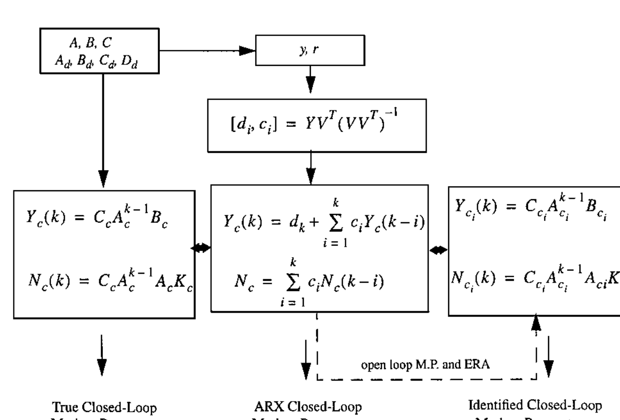

Fig. 2 Comparison of the closed-loop Markov parameters.

Table 1 Error percentage of the closed-loop system Markov parameters

Noise, %	$Y^c_{an} - Y^c_{id}$, %	$Y^c_{an} - Y^c_{arx}$, %	$Y^c_{arx} - Y^c_{id}$, %
0.01	0.7156	0.7631	0.4125
1.0	6.7190	7.5804	3.3978
5.0	12.507	13.629	6.8302
10.0	17.756	19.275	9.4349
15.0	24.655	26.647	11.847
20.0	27.262	30.700	13.033

Table 2 Error percentage of the open-loop system Markov parameters

Noise, %	$Y_{an} - Y_{arx}$, %	$Y_{an} - Y_{id}$, %	$Y_{arx} - Y_{id}$, %
0.01	0.4074	0.4085	0.0303
1.0	3.0263	3.0134	0.2441
5.0	5.1982	5.4720	0.5036
10.0	7.2277	7.1515	0.7069
15.0	10.913	11.275	0.8563
20.0	24.169	25.790	0.8967

Simulations using an ARX model order of 14 provide good results for moderate to high noise levels. The determination of the data length is performed in an analog fashion, and, while the ARX model order is kept at 14, the noise level is varied. The data length is set to be 5000. To compare the closed-loop Markov parameters, six different noise levels are used. The result shown in Table 1 are average values of 20 simulation runs for the Large Angle Magnetic Suspension Test Facility. The error between the analytical (Y^c_{an}) and identified (Y^c_{id}) Markov parameters and the analytical and the closed-loop Markov parameter derived from the estimated ARX model coefficients Y^c_{arx} are almost the same. However, the last column shows that the error is significantly lower for the ARX-model Markov parameters and the identified closed-loop Markov parameters. The error of Y^c_{an} and Y^c_{id} is slightly smaller for all noise cases than the error of Y^c_{an} and Y^c_{arx}. This must be associated with the situation in which during the ERA a set of singular values are selected, based on their magnitude, and the remaining set of singular values are truncated. The truncated singular values are associated with noise and, therefore, ERA can be regarded as a filter.

The calculated error percentage of the open-loop system Markov parameters are shown in Table 2. The result indicates the same trend found in Table 1. It is concluded that a major contribution of the error in the system identification process is introduced by the approximated system description of the ARX model. The error developed during the process of computing the identified system matrices from the identified open-loop Markov parameters is negligibly small, even with high measurement and process noise levels. The effect of ERA to the open-loop system Markov parameter errors is not reflected for the cases with higher noise values. The effects of ERA to the accuracy of the system description and the different values of error in the closed-loop and open-loop Markov parameters will be the subject of future studies.

Residual Whitening

In this section, the residual whitening method is introduced to reduce the model error generated by the approximated system description of the ARX model. First, Eq. (7) is arranged to reduce the ARX model order number. A new model (ARMAX) for describing the relationship between input and output is proposed. Then we use the minimization of the least-squares error of the residual term for whitening. After whitening the residual of the closed-loop ARMAX model parameters, one can recover the open-loop system Markov parameters.

To reduce the required model order number, the term $M\boldsymbol{y}_k$ is added and subtracted to the right-hand side of Eq. (7) to yield

$$\begin{aligned}\eta_{k+1} &= A_c\eta_k + B_c r_k + A_c K_c \varepsilon_k + M\boldsymbol{y}_k - M\boldsymbol{y}_k\\ &= A_c\eta_k + B_c r_k + A_c K_c\varepsilon_k + M(C_c\eta_k + \varepsilon_k) - M\boldsymbol{y}_k\\ &= (A_c + MC_c)\eta_k + (A_cK_c + M)\varepsilon_k + B_c r_k - M\boldsymbol{y}_k \end{aligned} \tag{17}$$

and

$$\boldsymbol{y}_k = C_c\eta_k + \varepsilon_k \tag{18}$$

Now the new relationship of reference input and output becomes

$$\boldsymbol{y}_k = \sum_{i=1}^{\infty} C_c\tilde{A}_c^{i-1}(-M)\boldsymbol{y}_{k-i} + \sum_{i=1}^{\infty} C_c\tilde{A}_c^{i-1}B_c r_{k-i} + \sum_{i=1}^{\infty} C_c\tilde{A}_c^{i-1}\bar{M}\varepsilon_{k-i} + \varepsilon_k \tag{19}$$

where

$$\tilde{A}_c = A_c + MC_c, \qquad \bar{M} = M + A_cK_c \tag{20}$$

Make $\tilde{A}_c$ asymptotically stable, so that $\tilde{A}_c^{i-1} \approx 0$ if $i \geq p$ for a sufficient large number p. Then Eq. (19) becomes

$$\boldsymbol{y}_k = \sum_{i=1}^{p} h_i\boldsymbol{y}_{k-i} + \sum_{i=1}^{p} t_i r_{k-i} + \sum_{i=1}^{p} s_i\varepsilon_{k-i} + \varepsilon_k \tag{21}$$

where

$$h_i = C_c\tilde{A}_c^{i-1}(-M), \qquad t_i = C_c\tilde{A}_c^{i-1}B_c, \qquad s_i = C_c\tilde{A}_c^{i-1}\bar{M}$$
$$i = 1, 2, \ldots, p$$

Equation (21) is an ARMAX model containing the dynamics of residual that is different from Eq. (9).

In the case of the open-loop system, one can write[6] the relationship of input and output as

$$\boldsymbol{y}_k = \sum_{i=1}^{\infty} C\bar{A}^{i-1}(-M_o)\boldsymbol{y}_{k-i} + \sum_{i=1}^{\infty} C\bar{A}^{i-1}B\boldsymbol{u}_{k-i} + \sum_{i=1}^{\infty} C\bar{A}^{i-1}\bar{M}_o\varepsilon_{k-i} + \varepsilon_k \tag{22}$$

where

$$\bar{A} = A + M_oC, \qquad \bar{M}_o = M_o + AK$$

Similarly, make $\bar{A}$ asymptotically stable, so that $\bar{A}^{i-1} \approx 0$ if $i \geq p_o$ for a sufficient large number p_o. Then Eq. (22) becomes

$$\boldsymbol{y}_k = \sum_{i=1}^{p_o} h_{oi}\boldsymbol{y}_{k-i} + \sum_{i=1}^{p_o} t_{oi}\boldsymbol{u}_{k-i} + \sum_{i=1}^{p_o} s_{oi}\varepsilon_{k-i} + \varepsilon_k \tag{23}$$

where

$$h_{oi} = C\bar{A}^{i-1}(-M_o), \qquad t_{oi} = C\bar{A}^{i-1}B, \qquad s_{oi} = C\bar{A}^{i-1}\bar{M}_o$$
$$i = 1, 2, \ldots, p_o$$

Equation (23) is the ARMAX model of the open-loop system. When looking at Eqs. (21) and (23), one can see the difference of the coefficients of the ARMAX models in both open-loop and closed-loop cases.

As stated in Eq. (19), $\tilde{A}_c = A_c + MC_c$, and the matrix M is used to make the new system (17) and (18) more stable than the original one in Eqs. (7) and (8). It can especially be used to reduce the requirement of ARMAX model order p. In the original system (7) and (8), K_c is working as an observer gain that includes Kalman filter gain K. We do not want to lose the information about the Kalman filter by making the observer equation more stable than the filter. The matrix M, however, could be chosen such that $\tilde{A}_c = A_c + MC_c$ is as stable as it could be without losing the Kalman filter information. In other words, we can choose an appropriate matrix M to limit the ARMAX model order number which need not be several times larger than the true order of the system.

Suppose there are N data points of $\boldsymbol{y}_k$ and r_k given, $k = 0, 1, \ldots, N-1$. Define

$$\theta = [t_1 \quad h_1 \quad \cdots \quad t_p \quad h_p], \qquad \Psi = [s_1 \quad s_2 \quad \cdots \quad s_p]$$

$$W = \begin{bmatrix} 0 & \varepsilon_0 & \varepsilon_1 & \cdots & \varepsilon_{p-1} & \cdots & \varepsilon_{N-2} \\ 0 & 0 & \varepsilon_0 & \cdots & \varepsilon_{p-2} & \cdots & \varepsilon_{N-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & \varepsilon_0 & \cdots & \varepsilon_{N-p-1} \end{bmatrix}$$

$$\boldsymbol{R} = [\varepsilon_0 \quad \varepsilon_1 \quad \cdots \quad \varepsilon_{N-1}]$$

Then Eq. (21) could be written in a matrix form

$$Y = \theta V + \Psi W + \boldsymbol{R} \tag{24}$$

where Y and V are the same as in Eq. (11).

The residual vector $\boldsymbol{R}$ equals

$$\boldsymbol{R} = Y - \theta V - \Psi W$$

$$= Y - [\theta \quad \Psi] \begin{bmatrix} V \\ W \end{bmatrix} \tag{25}$$

The cost function

$$J = \sum_{k=0}^{N-1} \varepsilon_k^T \varepsilon_k = \mathrm{tr}(\boldsymbol{R}\boldsymbol{R}^T) \tag{26}$$

Assuming the residual matrix W is available at this moment, then the least-squares solution of the parameters $\hat{\theta}$ and $\hat{\Psi}$ that minimize the cost function J is

$$[\hat{\theta} \quad \hat{\Psi}] = Y \begin{bmatrix} V \\ W \end{bmatrix}^+ = Y[V^T \quad W^T] \begin{bmatrix} VV^T & VW^T \\ WV^T & WW^T \end{bmatrix}^- \tag{27}$$

One can derive the parameter $\hat{\theta}$ to be a sum of ordinary least squares indirect closed-loop identification and a bias term.[6] Hence, one can write

$$\hat{\theta} = \theta^{\mathrm{LS}} - \theta^{\mathrm{bias}} \tag{28}$$

$$\hat{\Psi} = (Y - \theta^{\mathrm{LS}} V) W^T [WW^T - WV^T(VV^T)^{-1} VW^T]^- \tag{29}$$

where

$$\theta^{\mathrm{LS}} = YV^T(VV^T)^{-1}, \qquad \theta^{\mathrm{bias}} = \hat{\Psi} WV^T(VV^T)^{-1}$$

Following is the iteration procedure for whitening the residual.

1) Estimate the initial coefficient parameters θ of the closed-loop ARMAX model θ^{LS}. The other parameter Ψ is assumed to be zero in the beginning:

$$\hat{\theta}^{\mathrm{LS}} = YV^T(VV^T)^{-1}, \qquad \hat{\Psi}_0 = 0$$

2) Compute the initial residual sequence $\hat{R}_0$

$$\hat{R}_0 = Y - \hat{\theta}^{\mathrm{LS}} V$$

3) Compute $\hat{W}_0$, which is a function of $\hat{R}_0$, $\hat{W}_0 = \hat{W}(\hat{R}_0)$. Update $\hat{\Psi}_1$ and $\hat{\theta}_1$ by the following equations:

$$\hat{\Psi}_1 = (Y - \hat{\theta}^{\mathrm{LS}} V)\hat{W}_0^T \left[\hat{W}_0 \hat{W}_0^T - \hat{W}_0 V^T (VV^T)^{-1} V \hat{W}_0^T\right]^{-1}$$

$$\hat{\theta}_1 = \theta^{\mathrm{LS}} - \hat{\theta}_1^{\mathrm{bias}} = \theta - \hat{\Psi}_1 \hat{W}_0 V^T (VV^T)^{-1}$$

4) Generate a new sequence of $\hat{R}_1$:

$$\hat{R}_1 = Y - \hat{\theta}_1 V - \hat{\Psi}_1 \hat{W}_0$$

5) Iterate the procedure from step 3 to step 4 by generating the new residual matrix $\hat{W}_1$ and using the updated parameters $\hat{\theta}_1$ and $\hat{\Psi}_1$, so that the next cycle is calculated as follows, for example,

$$\hat{\Psi}_2 = (Y - \hat{\theta}^{\mathrm{LS}})\hat{W}_1^T \left[\hat{W}_1 \hat{W}_1^T - \hat{W}_1 V^T (VV^T) V \hat{W}_1^T\right]^-$$

and

$$\hat{\theta}_2^{\mathrm{bias}} = \hat{\Psi}_2 \hat{W}_1 V^T (VV^T)^{-1}, \qquad \hat{\theta}_2 = \hat{\theta}^{\mathrm{LS}} - \hat{\theta}_2^{\mathrm{bias}}$$

Recover System Matrices and Kalman Filter Gain

After having obtained the ARMAX model estimated parameters θ and Ψ, one can use the estimated coefficients to construct the closed-loop system Markov parameters $Y_c(k) = C_c A_c^{k-1} B_c$, Kalman filter Markov parameters $N_c(k) = C_c A_c^{k-1} A_c K_c$, and $Y_{mc}(k) = C_c A_c^{k-1} M$, $k = 1, 2, \ldots, p, p+1, \ldots$. Note that $Y_c(0) = 0$, $N_c(0) = I$, and $t_i = h_i = s_i = 0$, when $i > p$, where p is the ARMAX model order and where t_i, h_i, and s_i are the estimated ARMAX model parameters.

Remember that

$$\theta = [t_1 \quad h_1 \quad \cdots \quad t_p \quad h_p], \qquad \Psi = [s_1 \quad s_2 \quad \cdots \quad s_p]$$

$$\tilde{A}_c = A_c + MC_c, \qquad \bar{M} = M + A_c K_c, \qquad h_i = C_c \tilde{A}_c^{i-1}(-M)$$

$$t_i = C_c \tilde{A}_c^{i-1} B_c, \qquad s_i = C_c \tilde{A}_c^{i-1} \bar{M}, \qquad i = 1, 2, \ldots, p$$

One can first obtain the Markov parameters sequence $Y_c(k) = C_c A_c^{k-1} B_c$ and $Y_{mc}(k) = C_c A_c^{k-1} M$ by following

$$Y_c(k) = t_k + \sum_{i=1}^{k} h_i Y_c(k-i) \tag{30}$$

$$Y_{mc}(k) = -h_k + \sum_{i=1}^{k-1} h_i Y_{mc}(k-i) \tag{31}$$

Next, for the closed-loop Kalman filter Markov parameters $N_c = C_c A_c^{k-1} A_c K_c$, it is derived from both estimated parameters θ and Ψ. Note that

$$h_1 + s_1 = C_c(-M) + C_c(M + A_c K_c) = C_c A_c K_c$$

which is equal to $N_c(1)$

$$N_c(1) = h_1 + s_1 \tag{32}$$

To obtain $N_c(2)$, one can use

$$h_2 + s_2 = C_c(A_c + MC_c)A_c K_c = N_c(2) - h_1 N_c(1)$$

to yield

$$N_c(2) = h_2 + s_2 + h_1 N_c(1) \tag{33}$$

Similarly, one can find $N_c(3)$ by following

$$h_3 + s_3 = C_c A_c^2 A_c K_c + C_c M C_c A_c A_c K_c + C_c \tilde{A}_c M C_c A_c K_c$$

$$= N_c(3) - h_1 N_c(2) - h_2 N_c(1)$$

to yield

$$N_c(3) = h_3 + s_3 + h_1 N_c(2) + h_2 N_c(1) \tag{34}$$

By induction, one can show that the sequence $N_c(k)$ equals

$$N_c(k) = h_k + s_k + \sum_{i=1}^{k-1} h_i N_c(k-i) \tag{35}$$

Then, by using the closed-loop system Markov parameters $Y_c(k)$, the Kalman filter Markov parameters $N_c(k)$, and the known controller Markov parameters $Y_d(k)$, one can derive the open-loop system Markov parameters $Y(k) = CA^{k-1}B$ and Kalman filter Markov parameters $N(k) = CA^{k-1}AK$ from Eqs. (14) and (15).

Numerical Simulations

In this section, the large-gap magnetic suspension system is used as an example for whitening residual. The system is described in Ref. 5 and the discrete-time system model is shown in the Appendix. In numerical simulations, the model order number for both ARX (for closed-loop system identification) and ARMAX (for residual whitening models) is evaluated at 7 and 15, and the process and measurement noise (variance) varies from 0.01 to 20%. The number of data points is 5000. After the nonwhitening closed-loop system identification (CLID) and its residual whitening (CLID/RW) are performed, both sets of the open-loop system Markov parameters of the identified models are reconstructed. To evaluate the accuracy of the identified result, the reconstructed Markov parameters are compared with the true ones. Because there are five inputs and five outputs in this system, each open-loop system Markov parameter is a 5×5 matrix. To compare two matrices, F norm is used as shown in Eq. (16). Tables 3 and 4 show the error of open-loop system Markov parameters of nonwhitening one (Y^{CLID}) and whitening one ($Y^{\mathrm{CLID/RW}}$) compared with the true one (Y_{an}). The CLID/RW solution is obtained after four cycles of iteration. Figures 3–6 show the plots of convergence of whitened residual norm and autocorrelation of the whitened residual. In Table 3, the residual whitening method has a better result than the one without whitening when the model order equals 7. In Table 4, however, there is no difference between these two methods when the model order is 15. For a longer model order, because the original residual is almost whitened, one cannot improve the result by residual whitening. From the simulation results, the residuals can be quickly whitened after one iteration. This fast convergence rate is also observed in open-loop systems.[6] However, as compared with the open-loop systems,[6] the residual whitening process does not improve the residual norm too much. It seems that accuracy improvement is mainly due to the exploitation of the ARMAX model for the closed-loop systems. Further study on the convergence rate and the effects of the ARMAX model and residual whitening for both open-loop and closed-loop systems will be interesting.

Table 3 Error percentage of the open-loop system Markov parameters[a]

Noise (variance), %	$Y_{\mathrm{an}} - Y^{\mathrm{CLID}}$, %	$Y_{\mathrm{an}} - Y^{\mathrm{CLID/RW}}$, %
0.01	1.0334	1.0234
1	15.4462	9.9266
5	34.4472	14.3700
10	44.7847	23.7707
15	49.2913	34.8206
20	56.5067	38.4626

[a]Model order for ARX or ARMAX = 7.

Table 4 Error percentage of the open-loop system Markov parameters[a]

Noise (variance), %	$Y_{\mathrm{an}} - Y^{\mathrm{CLID}}$, %	$Y_{\mathrm{an}} - Y^{\mathrm{CLID/RW}}$, %
0.01	0.9066	1.0250
1	8.8467	9.5389
5	12.3540	13.0617
10	21.4697	21.0527
15	29.3945	30.8258
20	30.9603	30.8680

[a]Model order for ARX or ARMAX = 15.

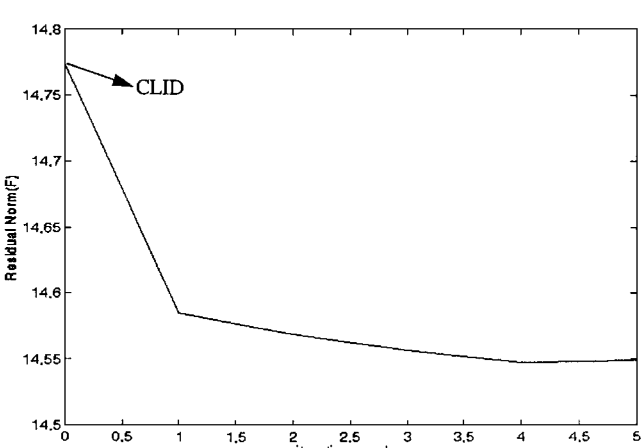

Fig. 3 Convergence of whitened residual norm for ARX or ARMAX = 7.

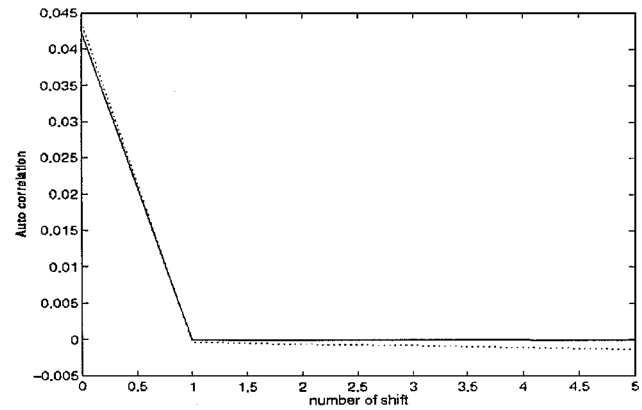

Fig. 4 Autocorrelation of whitened residual for ARX or ARMAX = 7. ⋯, CLID and ——, CLID/rw.

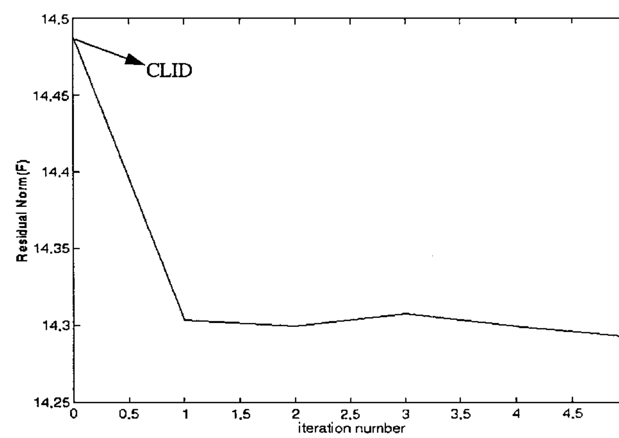

Fig. 5 Convergence of whitened residual norm for ARX or ARMAX = 15.

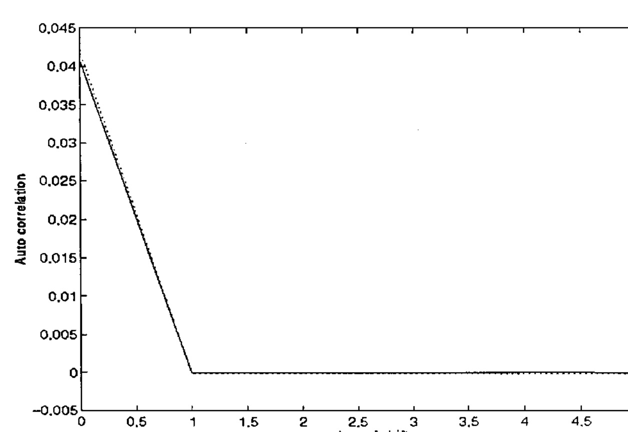

Fig. 6 Autocorrelation of whitened residual for ARX or ARMAX = 15. ⋯, CLID and ——, CLID/rw.

Conclusions

It has been shown that the accuracy of the system model during the identification process is related to the model error resulted from the approximated finite difference model. For a stochastic, time-invariant, closed loop system, a method for residual whitening has been introduced. The residual whitening of the auto-regressive moving average with exogenous input model improves the accuracy of the identified system model. Numerical simulation results show that for systems, which have to be modeled with a restricted number of model parameters, the residual whitening approach results in more accurate system identifications.

Appendix: State-Space Model of Large-Gap Magnetic Suspension System and Dynamic Feedback Controller

The discrete time state-space parameters of the large-gap magnetic suspension system are shown for sampling rate of 250 Hz. The finite-dimensional, linear, discrete-time, time invariant stochastic system can be expressed as

$$\boldsymbol{x}_{k+1} = A\boldsymbol{x}_k + B\boldsymbol{u}_k + w_k, \qquad \boldsymbol{y}_k = C\boldsymbol{x}_k + v_k$$

$$A = [A_{11} \quad A_{12}]$$

$$A_{11} = \begin{bmatrix} 1.1687 & 0.0006 & -0.0000 & 0.0000 & 0.0000 \\ -0.0000 & 1.1629 & -0.0000 & -0.0000 & -0.0000 \\ -0.0000 & 0.0001 & 1.0178 & -0.0017 & -0.0037 \\ -0.0000 & 0.0000 & 0.0001 & 1.0051 & 0.0001 \\ 0.0000 & 0.0002 & -0.0004 & 0.0008 & 1.0106 \\ 0.0000 & -0.0000 & -0.0021 & -0.0240 & 0.0005 \\ 0.0000 & -0.0001 & -0.0064 & -0.0001 & -0.0213 \\ -0.0000 & -0.0000 & 0.0109 & -0.0009 & -0.0045 \\ 0.0000 & -0.0000 & -0.0086 & 0.0009 & 0.0032 \\ 0.0000 & -0.0000 & 0.0004 & 0.0002 & 0.0006 \end{bmatrix}$$

$$A_{12} = \begin{bmatrix} 0.0000 & 0.0000 & -0.0000 & -0.0000 & -0.0633 \\ -0.0000 & 0.0000 & -0.0000 & 0.0000 & -0.0396 \\ 0.0021 & 0.0074 & -0.0127 & 0.0112 & 0.0254 \\ 0.0295 & 0.0006 & 0.0015 & -0.0011 & -0.0218 \\ -0.0018 & 0.0223 & 0.0066 & -0.0039 & -0.0242 \\ 0.9908 & 0.0028 & -0.0010 & 0.0003 & -0.0154 \\ -0.0041 & 0.9692 & 0.0064 & 0.0004 & -0.0066 \\ 0.0021 & 0.0050 & 0.9260 & -0.0549 & 0.0625 \\ 0.0009 & 0.0031 & -0.0589 & 0.9125 & 0.1245 \\ 0.0012 & 0.0545 & -0.0002 & -0.0002 & -0.1009 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.0035 & 0.0706 & 0.0519 & -0.0363 & -0.0633 \\ -0.0434 & -0.0326 & -0.0340 & -0.0425 & -0.0396 \\ 0.0580 & -0.0454 & 0.0983 & -0.0361 & 0.0254 \\ -0.0926 & -0.0315 & 0.0881 & 0.0865 & -0.0218 \\ 0.1160 & 0.0124 & 0.0263 & 0.0982 & -0.0242 \\ -0.1015 & -0.0368 & 0.1033 & 0.0854 & -0.0154 \\ 0.1373 & 0.0057 & 0.0719 & 0.0859 & -0.0066 \\ -0.0159 & -0.0637 & -0.1326 & 0.1165 & 0.0625 \\ 0.0158 & -0.1531 & -0.0261 & 0.0041 & 0.1245 \\ -0.0484 & -0.0800 & -0.0513 & -0.0553 & -0.1009 \end{bmatrix}$$

$$C = [C_{11} \quad C_{12}]$$

$$C_{11} = \begin{bmatrix} -0.0313 & 0.4029 & -0.0469 & 0.2269 & -0.0381 \\ 0.0291 & -0.4213 & 0.0006 & 0.2248 & 0.0290 \\ -0.4423 & 0.1071 & 0.1809 & 0.0553 & 0.0669 \\ -0.4254 & -0.1184 & -0.1787 & -0.0092 & -0.0829 \\ 0.4495 & -0.0763 & 0.0754 & 0.0273 & -0.1861 \\ 0.3889 & 0.1015 & -0.0614 & 0.0085 & 0.1739 \end{bmatrix}$$

$$C_{12} = \begin{bmatrix} -0.1961 & 0.1274 & -0.0363 & 0.0198 & -0.1513 \\ -0.2097 & -0.1079 & -0.0130 & 0.0297 & 0.1502 \\ -0.0618 & -0.0906 & -0.0418 & -0.2228 & -0.0472 \\ 0.0200 & 0.1217 & -0.2197 & -0.0559 & 0.0630 \\ -0.0400 & 0.1239 & 0.2109 & 0.0827 & 0.0464 \\ 0.0012 & -0.1277 & 0.0386 & 0.1913 & -0.0634 \end{bmatrix}$$

For simulation, the discrete-time state-space parameters of dynamics output feedback controller can be modeled as

$$p_{k+1} = A_d p_k + B_d \boldsymbol{y}_k, \qquad \boldsymbol{u}_k = C_d p_k + D_d \boldsymbol{y}_k + r_k$$

These matrices are

$$A_d = \begin{bmatrix} 0.3333 & 0 & 0 & 0 & 0 \\ 0 & 0.3333 & 0 & 0 & 0 \\ 0 & 0 & 0.6000 & 0 & 0 \\ 0 & 0 & 0 & 0.6000 & 0 \\ 0 & 0 & 0 & 0 & 0.6000 \end{bmatrix}$$

$$B_d = \begin{bmatrix} -0.0206 & 0.0206 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.0098 & -0.0098 & 0.0098 & 0.0098 \\ 0 & 0 & 0.0003 & -0.0003 & -0.0003 & 0.0003 \\ 0 & 0 & -0.0003 & 0.0003 & -0.0003 & 0.0003 \\ 0.0004 & 0.0004 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C_d = 1.0e+03 \times \begin{bmatrix} 0.0796 & 0.0000 & 7.3872 & 0.0000 & -5.5493 \\ 0.1032 & 0.0716 & -5.9772 & 4.3222 & -1.7160 \\ 0.0886 & 0.0442 & 2.2836 & -6.9917 & 4.4907 \\ 0.0886 & -0.0442 & 2.2836 & 6.9917 & 4.4907 \\ 0.1032 & -0.0716 & -5.9772 & -4.3222 & -1.7160 \end{bmatrix}$$

$$D_d = \begin{bmatrix} 10.8171 & 3.9903 & -7.0133 & 7.0133 & 7.0133 & -7.0133 \\ 6.7151 & -2.1362 & 11.2687 & -8.3349 & -3.0144 & 0.0807 \\ -2.1923 & -9.7904 & -7.9381 & 9.7505 & -5.4144 & 3.6020 \\ -2.1923 & -9.7904 & 3.6020 & -5.4144 & 9.7505 & -7.9381 \\ 6.7151 & -2.1362 & 0.0807 & -3.0144 & -8.3349 & 11.2687 \end{bmatrix}$$

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